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Spatially Varying Coefficients Models for Large Data with an Application to Real Estate Price Prediction

Swiss Real Estate Research Congress 2021 (online)

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20.05.2021

Hedonic Real Estate Pricing

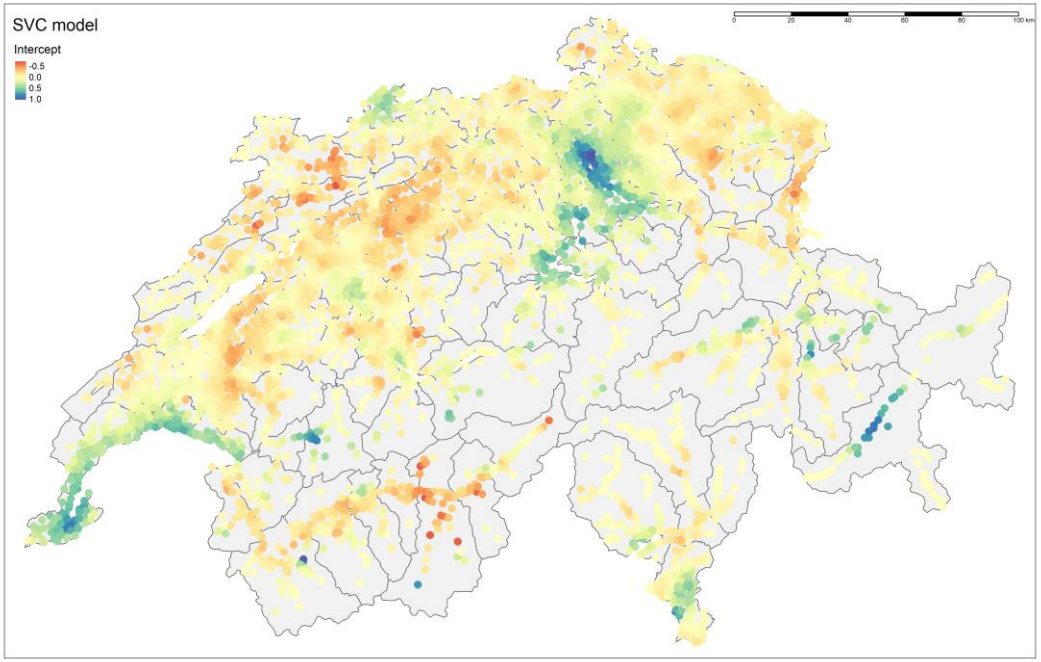
- Traditionally based on **linear regression model**

$$y_i = \beta^{(1)}(s_i) + \beta^{(2)}x_i^{(2)} + \ \ldots \ + \beta^{(p)}x_i^{(p)} + \epsilon_i$$

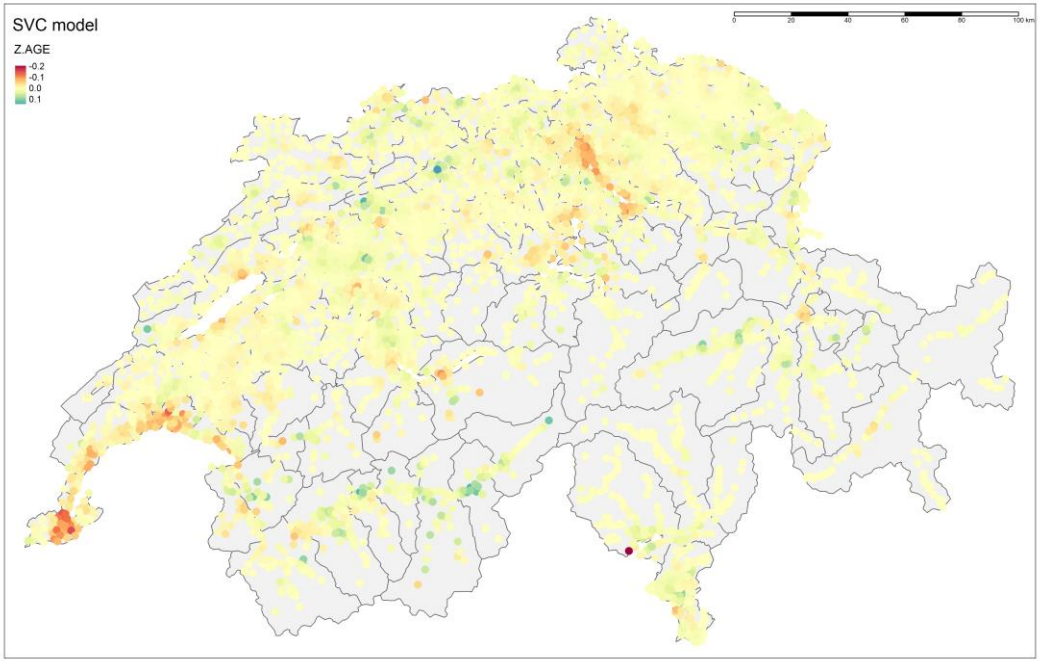
where
  - y_i: log(price) or log(rent)
  - $x_i^{(j)}$: properties of object i (size, age, standard, micro location, condition, etc.)
  - s_i: location of object i

- Strong conjecture from literature[6, 7] and domain experts that **coefficients $\beta^{(j)}$ are not constant over space**

Intercept (deviation from global mean)



Age Coefficient (deviation from global mean)



Spatially Varying Coefficients

- Linear regression model

$$y_i = \beta^{(1)}(s_i) + \beta^{(2)}x_i^{(2)} + \ \dots \ + \ \beta^{(p)}x_i^{(p)} + \epsilon_i$$

- Generalize to a **spatially varying coefficients (SVC) model**

$$y_i = \beta^{(1)}(s_i) + \beta^{(2)}(\boldsymbol{s}_i)x_i^{(2)} + \ \dots \ + \ \beta^{(p)}(\boldsymbol{s}_i)x_i^{(p)} + \epsilon_i$$

  - $\boldsymbol{s}_i$: location of observation i

Goal

- **Develop novel methodology for modelling of SVCs** that
  - **scales** in n (observations) and p (number of SVC)
  - is based on sound statistical methodology**: Gaussian processes** (GP)

- Innosuisse funded project in collaboration with Fahrländer Partner Raumentwicklung (FPRE), ZKB, BCV, and UBS

- Special thanks to **Jakob Dambon** (HSLU), **Reinhard Furrer** (UZH), **Manuel Lehner** (FPRE), **Jaron Schlesinger** (FPRE)

FP
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Existing Methods for Handling SVCs

Geographically Weighted Regression[1]

Very heuristic, bad properties

Bayesian SVC Processes[3]

$\boldsymbol{\beta}^{(j)}(\cdot) \sim N(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$
Gaussian processes

- Gaussian Predictive Processes[4]
- Gaussian Markov Random Field Approximation[5]

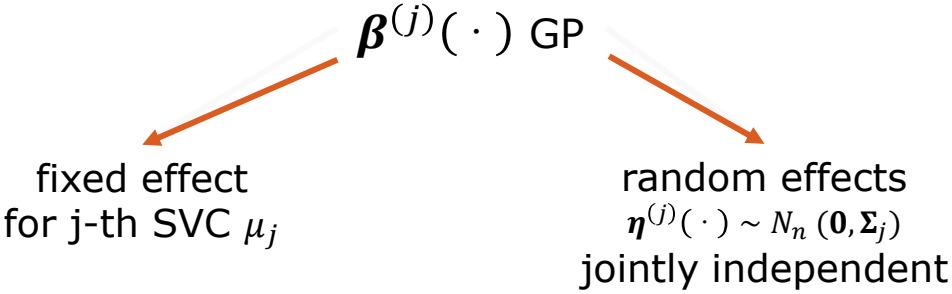
Existing model-based methods do not scale well

Eigenvector Spatial Filtering[2]

Not model-based, heuristic

Model-based Approach using Gaussian Processes (GP)

- Model SVCs $\boldsymbol{\beta}^{(j)}(\cdot)$ using Gaussian Processes (GP)



- Covariance matrix $\boldsymbol{\Sigma}_j$ parametrized by a range ρ_j and marginal variance σ_j^2

- Model in matrix notation

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\mu} + \boldsymbol{W}\boldsymbol{\eta} + \boldsymbol{\epsilon}$$

- Joint GP $\boldsymbol{\eta}(\cdot) \sim N_{np}(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\eta}} = \mathrm{diag}(\boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_\mathrm{p}))$ with according parametrization
  - $\boldsymbol{W}$ SVC data matrix and $\boldsymbol{X}$ fixed effect data matrix

Maximum Likelihood Estimation

- Log-likelihood: $\log L(\boldsymbol{y} \mid \boldsymbol{\theta}, \boldsymbol{\mu}) \propto \log\det \boldsymbol{\Sigma}_{\mathbf{y}} + (\boldsymbol{y} - \boldsymbol{X\mu})^{\top} \boldsymbol{\Sigma}_{\boldsymbol{y}}^{-1} (\boldsymbol{y} - \boldsymbol{X\mu})$

  where $\boldsymbol{\theta} = (\rho_1, \sigma_1^2, \ldots, \rho_p, \sigma_p^2, \tau^2)$ and $\boldsymbol{\Sigma}_{\mathbf{y}} = \boldsymbol{\Sigma}_{\boldsymbol{y}}(\boldsymbol{\theta}) = \boldsymbol{W\Sigma_{\eta}W}^{\top} + \tau^2 I_{n\times n}$

- Numeric optimization required. Challenges:
  - High-dimensional parameter space $(\boldsymbol{\theta}, \boldsymbol{\mu}) \in \boldsymbol{\Omega} = (0, \infty)^{2p+1} \times \mathrm{R}^{p}$
  - Computationally intensive tasks

Ad-hoc MLE does **not** scale in p nor n

Proposed Approach

- Use **profile likelihood** (generalized least squares estimate) for $\boldsymbol{\mu}$

$$\hat{\boldsymbol{\mu}} = \left(\boldsymbol{X}^\top \boldsymbol{\Sigma}_{\boldsymbol{y}}^{-1}(\hat{\boldsymbol{\theta}})\, \boldsymbol{X}\right)^{-1} \boldsymbol{X}^\top \boldsymbol{\Sigma}_{\boldsymbol{y}}^{-1}(\hat{\boldsymbol{\theta}})\, \boldsymbol{y}$$

Dimension reduction of parameter space

- Prior **independence assumption** on GPs

$$\boldsymbol{\Sigma}_{\boldsymbol{y}}(\boldsymbol{\theta}) = \sum_{j=1}^{p} \boldsymbol{\Sigma}_j \odot \left(\boldsymbol{x}^{(j)} (\boldsymbol{x}^{(j)})^\top\right) + \tau^2 I_{n\times n}$$

Efficient updating of $\boldsymbol{\Sigma}_{\boldsymbol{y}}(\boldsymbol{\theta})$

- **Covariance matrix tapering**[8]

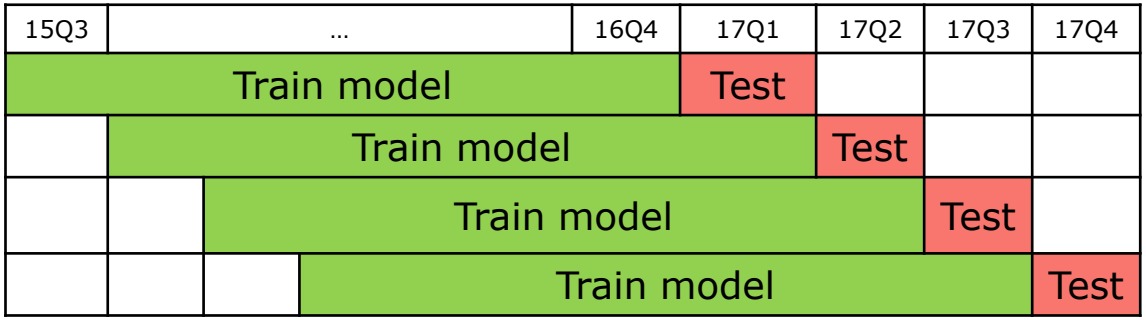
$$\boldsymbol{\Sigma}^{taper} \odot \boldsymbol{\Sigma}_{\boldsymbol{y}}(\boldsymbol{\theta}) = \boldsymbol{\Sigma}^{taper} \odot \left(\sum_{j=1}^{p} \boldsymbol{\Sigma}_j \odot \left(\boldsymbol{x}^{(j)} (\boldsymbol{x}^{(j)})^\top\right)\right) + \tau^2 I_{n\times n}$$

Sparse matrices to speed up Cholesky decomposition

- Further regularize likelihood using priors[9, 10]

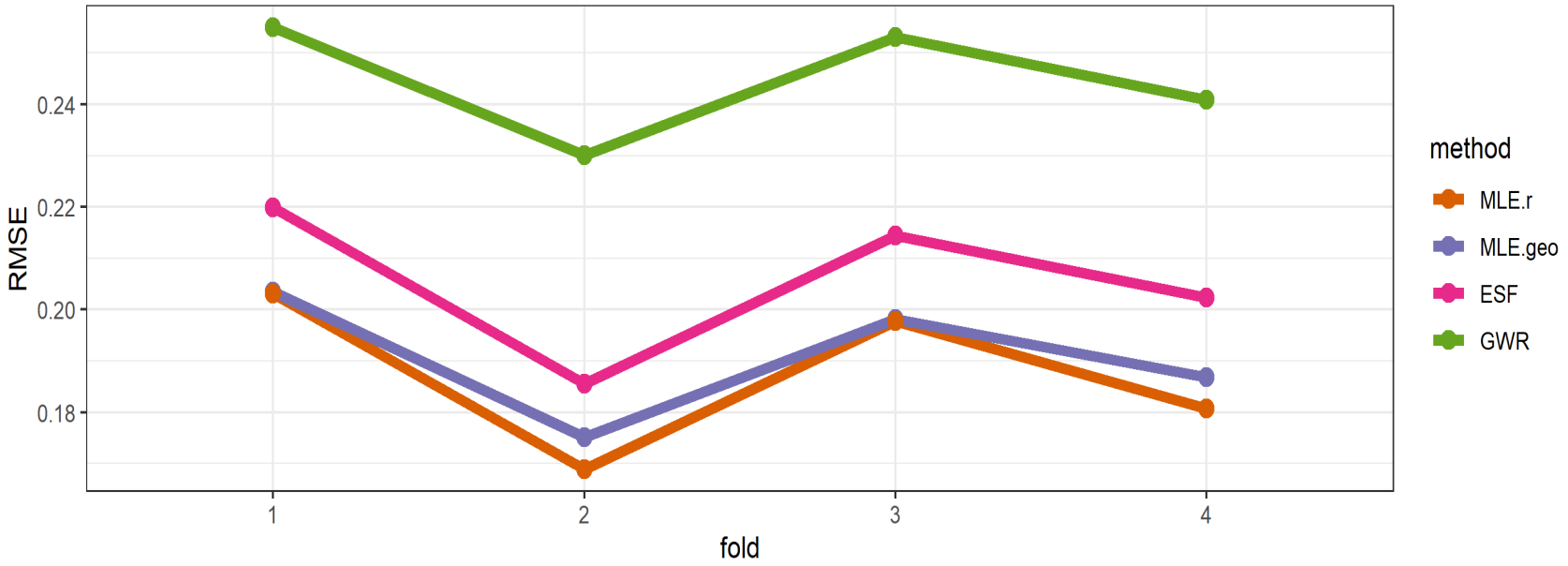
Comparison Using Cross-Validation

- Transaction prices for apartments in Switzerland
- Temporal "cross-validation" for model comparison

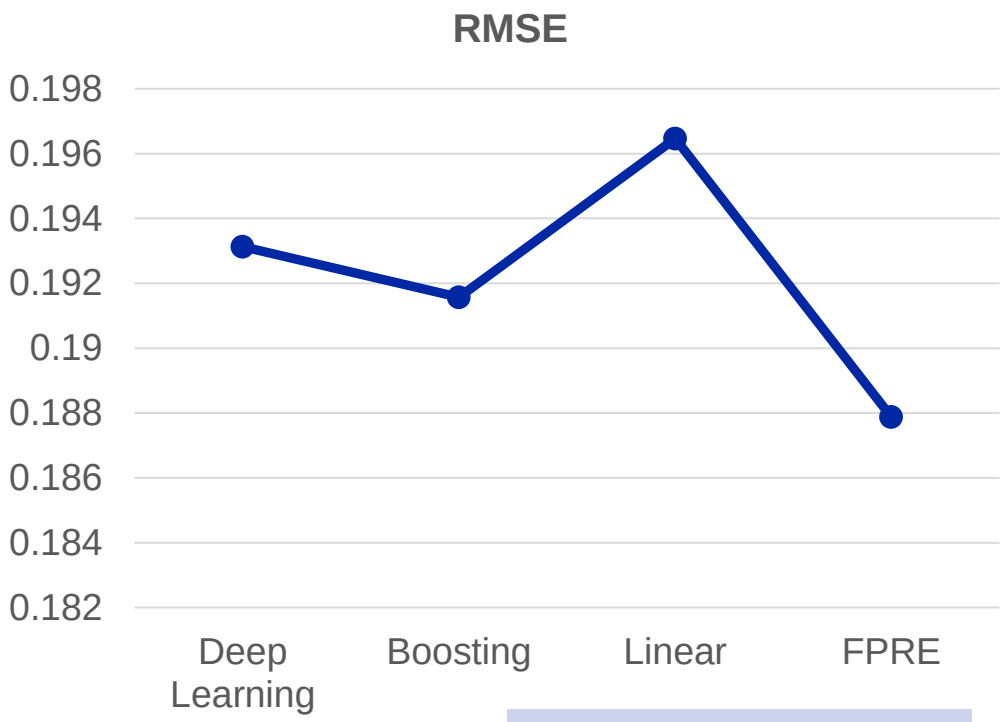


$p = 8$
$n \approx 15'000$

Out-of-sample RMSE



What About Machine Learning (ML) and Deep Learning?



Black-box ML is not the answer!

Comments

- **RMSE over 4 quarters**
- Models
  - **Linear**: linear model with spatial fixed effects (dummies for locations)
  - **FPRE**: linear model with spatial modelling used by FPRE (spline based but using additional historical data)
  - **Deep Learning**[11]: large amount of network architectures and tuning parameters considered
- *Note*: Slightly different data and linear model specifications compared to above results

Thank you for your attention!

- R-package **`varycoef`**

  https://github.com/jakobdambon/varycoef

- Article: Dambon, Sigrist, and Furrer (2021)[12]

References

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